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Aglianò–Montagna type decomposition of linear pseudo hoops and its applications

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Abstract

We decompose every linear pseudo hoop as an Aglianò–Montagna type of ordinal sum of linear Wajsberg pseudo hoops which are either negative cones of linear ℓ-groups or intervals in linear unital ℓ-groups with strong unit. We apply the decomposition to present a new proof that every linear pseudo BL-algebra and consequently every representable pseudo BL-algebra is good. Moreover, we show that every maximal filter and every value of a linear pseudo hoop is normal, and every σ-complete linear pseudo hoop is commutative.

MSC: 06D35; 03G12; 03B50

1. Introduction

The authors of [16] recently presented pseudo hoops which were originally introduced by Bosbach in [2,3] under the name‘ residuated integral monoids’. Pseudo hoops generalize pseudo BL-algebras [5,6], pseudo MV-algebras [15] (= GMV-algebras [22]) and bounded $R\ell$-monoids [10,11]. Pseudo MV-algebras are always intervals in unital ℓ-groups [7], i.e. of the form $[0, u]$, where u is a strong unit. These structures can also be studied in the frames of integral residuated lattices [13], more precisely, a pseudo hoop is a meet-semilattice ordered residuated, integral and divisible monoid.

We recall that some interesting connections to non-commutative logic using these structures can be found in [18,19]. For example, the paper [19] presents some results on the logic psBL (pseudo-basic fuzzy logic, the generalization of BL not assuming commutativity of conjunction) and on the analogous logic psMTL, a non-commutative version of the monoidal t-norm logic MTL of Esteva and Godo [12]. The logic psBL has its algebraic counterpart – a variety of algebras of truth functions – pseudo BL-algebras. By a pseudo t-norm is assumed a binary operation $*$ on $[0, 1]$ which is associative, non-decreasing in both arguments, and has 1 as a both-side neutral element. For example, let $0 < a < b < 1$ and let $x * y = 0$ if $x \leq a$ and $y \leq b$, and $x * y = \min(x, y)$ otherwise. This is a non-commutative pseudo t-norm which is left continuous in both arguments and has both residua; hence, it defines a psMTL-algebra. In addition, as a matter of further research the following problem was indicated – to find a common generalization of the hoop logic and psBL – and maybe pseudo hoops could be such a generalization.

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Hájek [17] showed that every saturated linear BL-algebra is an ordinal sum of irreducible linear pseudo BL-algebras. For linear pseudo BL-algebras this was generalized in [9]. Aglianò and Montagna [1] defined another type of ordinal sum and they stressed that this ordinal sum of a family of linear Wajsberg hoops is a fundamental construction in theory of BL-algebras and hoops because they proved that every totally ordered hoop is the ordinal type of a family of Wajsberg hoops.

We generalize this type of the ordinal sum, the Aglianò–Montagna ordinal sum type, to decompose every linear pseudo hoop as the ordinal sum of linear Wajsberg pseudo hoops. Thus linear Wajsberg pseudo hoops are basic building bricks for representable pseudo hoops, and every linear Wajsberg pseudo hoop is either the negative cone of a linearly ordered ℓ-group or the interval $[0, u]$ of a linearly ordered unital ℓ-group with strong unit u.

In addition, using the Aglianò–Montagna type decomposition, we present a new proof of the fact [9] that every linear pseudo BL-algebra and thus the variety of representable pseudo BL-algebra is a family of good pseudo BL-algebras, i.e., $x^{-\sim} = x^{\sim-}$ for any $x \in M$. This gives a partial answer to an open problem posed in [6, Problem 3.21] of whether there exists a pseudo BL-algebra which is not good, and we generalize this result. Moreover, we demonstrate the power of the decomposition method proving that a maximal filter of a linear pseudo hoop is normal.

In addition, we apply the Aglianò–Montagna type decomposition to show that every σ-complete linear pseudo hoop is commutative which gives a partial positive answer to the problem posed in [14] whether every σ-complete pseudo BL-algebra is commutative.

2. Pseudo BL-algebras and pseudo hoops

A *pseudo BL-algebra* was introduced in [5,6] as an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid (need not be commutative), i.e., $\odot$ is associative with neutral element 1.
(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;
(iii) $x \odot y \le z$ iff $x \le y \to z$ iff $y \le x \rightsquigarrow z$ $x, y \in M$;
(iv) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x),\ x, y \in M$;
(v) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow y),\ x, y \in M$.

We recall that $\wedge$, $\vee$ and $\odot$ have higher priority than $\to$ or $\rightsquigarrow$, and M is a distributive lattice.

We say that a pseudo BL-algebra M is a *BL-algebra* if $x \odot y = y \odot x$ for all $x, y \in M$. This is equivalent to the statement that $\to = \rightsquigarrow$.

Let M be a pseudo BL-algebra. Let us define two unary operations (negations) $^-$ and $^\sim$ on M such that $x^- := x \to 0$ and $x^\sim := x \rightsquigarrow 0$ for any $x \in M$. It is easy to show that

$$x \odot y = 0 \Leftrightarrow y \le x^\sim \Leftrightarrow x \le y^-.$$

The basic properties of pseudo BL-algebras were studied in [5,6].

Pseudo MV-algebras, or equivalently *GMV-algebras*, were introduced in [15] and [22], respectively. It is possible to show that a pseudo BL-algebra M is a GMV-algebra, [5, Proposition 3.27], if and only if M satisfies the identities $x^{-\sim} = x = x^{\sim-}$. We recall that in [13] GMV-algebras have a different meaning: in fact GMV-algebras in this sense are the direct product of an ℓ-group and a Wajsberg pseudo hoop in our sense. GMV-algebras in the sense of [15] and [22] are precisely the bounded and integral GMV-algebras in the sense of [13].

Another generalization of pseudo BL-algebras was recently made in [16], where the authors introduced pseudo hoops, which were originally introduced by Bosbach in [2,3] under the name ‘residuated integral monoids’. We recall that a *pseudo hoop* is an algebra $(M; \odot, \to, \rightsquigarrow, 1)$ of type $\langle 2, 2, 2, 0\rangle$ such that, for all $x, y, z \in M$,

(i) $x \odot 1 = x = 1 \odot x$;
(ii) $x \to x = 1 = x \rightsquigarrow x$;
(iii) $(x \odot y) \to z = x \to (y \to z)$;
(iv) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
(v) $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

If $\odot$ is commutative (equivalently $\to = \rightsquigarrow$), M is said to be a *hoop*. If we set $x \leq y$ iff $x \to y = 1$ (this is equivalent to $x \rightsquigarrow y = 1$), then $\leq$ is a partial order such that $x \wedge y = (x \to y) \odot x$. A pseudo hoop M is *bounded* if there is an element $0 \in M$ such that $0 \leq x$ for each $x \in M$. If there is no least element, M is said to be *unbounded*. Every pseudo BL-algebra is a bounded pseudo hoop.

For any $x \in M$ and for any integer $n \geq 0$, we define x^n inductively: $x^0 := 1$ and $x^n := x^{n-1} \odot x$ for $n \geq 1$.

A subset F of a pseudo hoop is said to be a *filter* if (i) $x, y \in F$ implies $x \odot y \in F$, and (ii) $x \leq y$ and $x \in F$ imply $y \in F$. We recall that normal filters are in a one-to-one correspondence with congruences.

An element of $a \in M$ is an *idempotent* if $a \odot a = a$.

Basic properties of pseudo hoops are studied in [16].

A pseudo hoop M is said to be *Wajsberg* if, for all $x, y \in M$,

(W1) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$;
(W2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

Pseudo Wajsberg hoops correspond exactly to integral GMV-algebras in the terminology of [13]. However, the lattice operations in GMV-algebras are primitive, but not primitive for Wajsberg pseudo hoops, they are definable there.

A bounded pseudo Wajsberg hoop $(M; \odot, \to, \rightsquigarrow, 0, 1)$ is said to be a *pseudo Wajsberg algebra* [4]; for them we define $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$; in addition, $0 = 1^- = 1^\sim$. By [4], the variety of pseudo Wajsberg algebras is term-equivalent with the variety of pseudo MV-algebras.

A pseudo hoop M is said to be *basic* if, for all $x, y, z \in M$,

(B1) $(x \to y) \to z \leq ((y \to x) \to z) \to z$;
(B2) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

It is straightforward to verify that any linearly ordered pseudo hoop and hence any representable pseudo hoop is basic. We recall that not every pseudo hoop is basic, [16, Remark 5.10] [take M_1 a nonlinear pseudo hoop and $M_2 = 2^1$, then the ordinal sum $M = M_1 \bigoplus M_2$ (for definition see below) is not basic], and not all pseudo BL-algebras are representable (take e.g. a GMV-algebra $[0, u]$ of a non-representable ℓ-group).

By [16, Proposition 4.6], basic pseudo hoops are precisely those pseudo hoops which satisfy (v) and have a distributive lattice reduct. Any pseudo Wajsberg hoop is a basic pseudo hoop [16, Proposition 4.9], and the variety of pseudo BL-algebras coincides with the variety of bounded basic pseudo-hoops. (We recall lattice operations are primitive in pseudo BL-algebras and not in bounded basic pseudo hoops, but they are definable there.)

Example 2.1. (1) Let $(G; \cdot, \vee, \wedge, ^{-1}, e)$ be an ℓ-group written multiplicatively. For the negative cone $G^- = \{g \in G : g \leq e\}$ we define $a \odot b = a \cdot b$, $a \to b = (b \cdot a^{-1}) \wedge e$, and $a \rightsquigarrow b = (a^{-1} \cdot b) \wedge e$. Then $(G^-; \odot, \to, \wedge, e)$ is a basic unbounded cancellative pseudo hoop.

(2) Let $(G; +, \vee, \wedge, -, 0)$ be an ℓ-group written additively. Choose $u \geq 0$ and we endow the interval $[0, u]$ with $x \odot y = (x - u + y) \vee 0$, $x \to y = (y - x + u) \wedge u$, $x \rightsquigarrow y = (u - x + y) \wedge u$, $x, y \in [0, u]$. Then $([0, u]; \odot, \to, \rightsquigarrow, u)$ is a bounded pseudo hoop with the least element 0 which is term-equivalent with a pseudo MV-algebra.

(3) Let $(G^-; \odot, \to, \wedge, e)$ be an unbounded cancellative pseudo hoop. The ordinal sum of the Boolean algebra 2^1 with G^- gives a pseudo BL-algebra which is a pseudo-product algebra [6, Example 2.21]; for the definition see [6, Definition 2.18].

The following lemma was originally proved for linear pseudo BL-algebras [6, Lemma 3.3], and in the same way it can be proved for linear pseudo hoops.

Lemma 2.2. *Let M be a linear pseudo hoop.*

(1) *If $x \leq u \leq y$ and $u \odot u = u$, then $x \odot y = x = y \odot x$.*
(2) *If $u \odot u = u$ and $x < u \leq y$, then $y \rightsquigarrow x = x = y \to x$.*

The proof of the following important lemma is identical to one for pseudo BL-algebras proved in [6, Proposition 3.5].

Lemma 2.3. *Let M be a linear pseudo hoop.*

(1) *There exist no elements* $x, y, z \in M$ *such that*

$$x < z < y, \quad x \odot y = x, \quad x \odot z < x, \quad y \odot z < z.$$

(2) *There exist no elements* $x, y, z \in M$ *such that*

$$x < z < y, \quad y \odot x = x, \quad z \odot x < x, \quad z \odot y < z.$$

The following lemma was proved in [9] for linear pseudo BL-algebras.

Lemma 2.4. *Let M be a linear hoop and let $x \odot x < x$. For $v \in M$, $v \odot x = x$ if and only if $x = x \odot v$. In such a case, $v \to x = x = v \rightsquigarrow x$.*

Proof. It is clear that $v \in Y_x := \{z \in M : z \odot x = x\}$ and $x \notin Y_x$. Hence, $x < v$ and $v \to x \notin Y_x$, otherwise $x = x \wedge v = (v \to x) \odot v \in Y_x$, a contradiction fact that Y_x is a filter.

Then $x \leq v \to x < v$. Assume $x < v \to x < v$. Then $v \odot x = x$, $(v \to x) \odot v = x < v \to x$; therefore by Lemma 2.3, $(v \to x) \odot x = x$ which yields $v \to x \in Y_x$ which is a contradiction. Therefore, our assumption was wrong, and $x = v \to x$.

Calculate $x = x \wedge v = (v \to x) \odot v = x \odot v$.

Similarly, we can prove that if $x = x \odot v$, then $v \rightsquigarrow x = x$ and $x = v \odot x$. $\square$

3. Ordinal sums and cuts

According to [1] we introduce an ordinal sum of pseudo hoops. Let $\{M_i : i \in I\}$ be a system of pseudo hoops with a linearly ordered index set $(I; \leq)$ such that $M_i \cap M_j = \{1\}$ for all $i \neq j$, $i, j \in I$. We set $M = \bigcup_{i \in I} M_i$ and on M we define the operation $\odot$, $\to$ and $\rightsquigarrow$ as follows

$$x \odot y = \begin{cases} x \odot_i y & \text{if } x, y \in M_i, \\ x & \text{if } x \in M_i \setminus \{1\},\ y \in M_j,\ i < j, \\ y & \text{if } x \in M_i,\ y \in M_j \setminus \{1\},\ i > j, \end{cases}$$

$$x \to y = \begin{cases} x \to_i y & \text{if } x, y \in M_i, \\ y & \text{if } x \in M_i,\ y \in M_j \setminus \{1\},\ i > j, \\ 1 & \text{if } x \in M_i \setminus \{1\},\ y \in M_j,\ i < j, \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} x \rightsquigarrow_i y & \text{if } x, y \in M_i, \\ y & \text{if } x \in M_i,\ y \in M_j \setminus \{1\},\ i > j, \\ 1 & \text{if } x \in M_i \setminus \{1\},\ y \in M_j,\ i < j. \end{cases}$$

Then M with $1, \odot, \to$ and $\rightsquigarrow$ is a pseudo hoop called the *ordinal sum* of $\{M_i : i \in I\}$. If I has a minimum, 0, and M_0 is a bounded pseudo hoop, then $\bigoplus_i M_i$ is bounded. If all M_is are linear so is $\bigoplus_i M_i$. If all M_is are linear pseudo hoops and M_0 is bounded, then $\bigoplus_i M_i$ is a linear pseudo BL-algebra.

A linearly ordered pseudo hoop is said to be *sum irreducible* (shortly irreducible) if it cannot be written as the ordinal sum of two linearly ordered non-trivial pseudo hoops.

A pair (X, Y) of subsets of a linear pseudo hoop M is said to be a *cut* of M if

(i) $X \cup Y = M$,
(ii) $x \leq y$ for all $x \in X$ and $y \in Y$,
(iii) Y is closed under $\odot$,
(iv) $x \odot y = x$ or $y \odot x = x$ for all $x \in X$ and $y \in Y$.

If (iv) is changed to

(v) $x \odot y = x = y \odot x$, for all $x \in X$ and $y \in Y$,

then (X, Y) is said to be a *strong cut*. In [9, Propositions 3.4, 3.11], there was proved that every cut of a pseudo BL-algebra is strong, and if $x, v \in M$, $x = v \odot x$ iff $x = x \odot v$; moreover $v \to x = x = v \rightsquigarrow x$.

The same is true for any cut (X, Y) of a linear pseudo hoop M. Indeed, if M is not bounded, let $B_2 = \{0, 1\}$ be the two-element Boolean algebra. Then $\hat{M} = B_2 \bigoplus M$ is a linear pseudo BL-algebra and $(X \cup \{0\}, Y)$ is a cut of

$\hat{M}$ which is strong. Therefore, (X, Y) is strong, too. In addition, if $x, v \in M$, $x = v \odot x$ iff $x = x \odot v$; moreover $v \to x = x = v \rightsquigarrow x$.

If (X, Y) is a cut of M, then either $X \cap Y = \emptyset$ or $X \cap Y$ consists of an idempotent singleton. In any case, $x \odot y = x = y \odot x$ for all $x \in X$, $y \in Y$, and $y \to x = x = y \rightsquigarrow x$ if $x \in X \setminus Y$ and $y \in Y \setminus X$.

If M is a linear pseudo BL-algebra, a cut (X, Y) is said to be *trivial* if $X = \{0\}$ or $Y = \{1\}$; if M is an unbounded linear pseudo hoop, a cut (X, Y) is said to be *trivial* if $X = \emptyset$ or $Y = \{1\}$.

Proposition 3.1. *Let (X, Y) be a cut of a linear pseudo hoop (linear pseudo BL-algebra) M such that $X \cap Y = \emptyset$. Then $X \cup \{1\}$ is a linear pseudo subhoop (linear pseudo BL-subalgebra) of M and Y is a linear pseudo subhoop of M with respect to $\odot$, $\to$, and $\rightsquigarrow$. Moreover, $M = (X \cup \{1\}) \bigoplus Y$. If (X, Y) is a non-trivial cut, M is not irreducible.*

Conversely, if $M = M_1 \bigoplus M_2$, where M_1 is a linear pseudo hoop (pseudo BL-algebra), and M_2 is a linear pseudo hoop, then $(M_1 \setminus \{1\}, M_2)$ is a cut of M. If M_1 and M_2 are non-trivial so is $(M_1 \setminus \{1\}, M_2)$.

Proof. Assume that (X, Y) is a cut. Then X is closed under $\odot$ because X is downwards closed, i.e., if $a \le b$ and $b \in X$, then $a \in X$ and $x \odot y, y \odot x \le y$. Moreover, if $x \in X$ and $y \in Y$, then $y \to x, y \rightsquigarrow x \in X$, therefore $x = y \odot (y \rightsquigarrow x) = y \rightsquigarrow x$, similarly $x = (y \to x) \odot y = y \to x$ while Y is closed under $\odot$.

Assume $x_1, x_2 \in X$ and $x_1 > x_2$. Then $x_1 \to x_2 \in X$. Suppose the converse, i.e., $x_1 \to x_2 \in Y$, therefore, $x_2 = x_1 \wedge x_2 = (x_1 \to x_2) \odot x_1 = x_1$ which is a contradiction. Similarly $x_1 \rightsquigarrow x_2 \in X$.

Consequently, $M_1 := X \cup \{1\}$ is a linear pseudo subhoop (pseudo BL-subalgebra) of M.

Since X is downwards directed, Y is upwards directed and Y is a filter. Therefore Y is closed with respect to $\odot$, $\to$, and $\rightsquigarrow$, and $M = M_1 \bigoplus M_2$, where $M_2 = Y$, and Y is a pseudo subhoop of M.

The rest of the statement is evident. □

Proposition 3.2. *Let M be a linear pseudo hoop and let m and a be elements of $M \setminus \{1\}$ such that $m \to a = a$. Set $X_m^{\to} = \{x \in M \setminus \{1\} : m \to x = x\}$, and $Y_m^{\to} = M \setminus X_m^{\to}$. Then $(X_m^{\to}, Y_m^{\to})$ is a non-trivial cut of M.*

Proof. (1) It is clear that $a \in X_m^{\to}$ and $m \in Y_m^{\to}$ and $a < m$. Then $a = a \wedge m = (m \to a) \odot m = a \odot m$.

(2) $X_m^{\to}$ is downwards closed. Indeed, let $b \le a$ and $a \in X_m^{\to}$. To prove that $b \in X_m^{\to}$, it is sufficient to verify $(m \to b) \to b = 1$. We recall $m \to b \le m \to a = a$. Hence,

$$\begin{aligned}(m \to b) \to b &= (a \wedge (m \to b)) \to b \\ &= ((a \to (m \to b)) \odot a) \to b \quad [5, \text{Proposition 3.8}] \\ &= (a \to (m \to b)) \to (a \to b) \\ &= ((a \odot m) \to b) \to (a \to b) = (a \to b) \to (a \to b) = 1.\end{aligned}$$

Consequently, $X_m^{\to}$ is closed under $\odot$.

Claim (3) *If $a, b \in X_m^{\to}$, $b < a$, then $a \rightsquigarrow b \in X_m^{\to}$.*

Check

$$m \to (a \rightsquigarrow b) = a \rightsquigarrow (m \to b) = a \rightsquigarrow b.$$

Claim (4) *Let $a \in X_m^{\to}$, $b \in Y_m^{\to}$, then $b \to a = a = a \odot b$.*

We have $b \le (b \to a) \rightsquigarrow a$, and since $X_m^{\to}$ is downwards closed and $b \notin X_m^{\to}$, we conclude $(b \to a) \rightsquigarrow a \notin X_m^{\to}$.

Calculate $m \to ((b \to a) \rightsquigarrow a) = (b \to a) \rightsquigarrow (m \to a) = (b \to a) \rightsquigarrow a$. Thus either $(b \to a) \rightsquigarrow a \in X_m^{\to}$ or $(b \to a) \rightsquigarrow a = 1$. From the above, we have only the second possibility, i.e., $a \le b \to a \le a$.

In addition, $a = a \wedge b = (b \to a) \odot b = a \odot b$.

Since $X_m^{\to}$ is downwards closed, $Y_m^{\to}$ is upwards closed. Hence, $Y_m^{\to}$ is closed under $\to$ and $\rightsquigarrow$. We state that $Y_m^{\to}$ is closed under $\odot$. If not, there exists $a, b \in Y_m^{\to}$ such that $a \odot b \in X_m^{\to}$. Then by (4), $b \to (a \odot b) = a \odot b$. But $a \wedge (b \to (a \odot b)) = (a \to (b \to (a \odot b))) \odot a = (a \odot b \to a \odot b) \odot a = a$, i.e., $a \le b \to (a \odot b) = a \odot b \in X_m^{\to}$ which yields $a \in X_m^{\to}$, a contradiction.

Due to [9, Proposition 3.11], $(X_m^{\to}, Y_m^{\to})$ is a cut. □

Proposition 3.3. *Let M be a linear pseudo hoop and let m and a be elements of $M \setminus \{1\}$ such that $m \rightsquigarrow a = a$. Set $X_m^{\rightsquigarrow} = \{x \in M \setminus \{1\} : m \rightsquigarrow x = x\}$, and $Y_m^{\rightsquigarrow} = M \setminus X_m^{\rightsquigarrow}$. Then $(X_m^{\rightsquigarrow}, Y_m^{\rightsquigarrow})$ is a non-trivial cut of M.*

Proof. This follows the same ideas as that of Proposition 3.2. □

Let $a \in X_m^{\to}$, then $a = a \odot m$. If a is not an idempotent of M, then by [9, Proposition 3.4] $a = m \odot a = m \rightsquigarrow a$ which gives $a \in X_m^{\rightsquigarrow}$ and vice versa.

Proposition 3.4. *Let M be a linear pseudo hoop and let a be an element of M which is not idempotent. Set $Y_a = \{z \in M : z \odot a = a\}$ and $X_a = M \setminus Y_a$. Then (X_a, Y_a) is a cut of M.*

Proof. It is clear that $a \in X_a$ and $1 \in Y_a$. To prove that (X_a, Y_a) is a cut, we verify the conditions (i)–(iv) of the definition of a cut.

(i) Clear. (ii) Let $z \in X_a$, $u \in Y_a$. If $u < z$, then $u \odot a \le z \odot a < a$ since $z \in X_a$ which contradicts $u \in Y_a$. Therefore, $z \le u$.

(iii) If $z_1, z_2 \in Y_a$, then $z_1 \odot z_2 \odot a = z_1 \odot a = a$ (Y_a is in fact a filter).

(iv) Let $u \in X_a$, $v \in Y_a$, and let $u \odot v < v$. Since $v \odot a = a$, there are two cases:

(a) $u \le a < v$. This yields $v \odot u = v \odot (u \wedge a) = v \odot (a \odot (a \rightsquigarrow u)) = a \odot (a \rightsquigarrow u) = a \wedge u = u$. On the other hand, using Lemma 2.3, we have $v \odot a = a$ which gives $u \odot v = (u \wedge a) \odot v = (a \to u) \odot a \odot v = (a \to u) \odot a = a \wedge u = u$.

(b) $a < u < v$. We have $v \odot a = a, u \odot a < a$ (since $u \notin Y_a$) and $u \odot v = u$ (while $u \odot v \le u$ and applying Lemma 2.2).

Similarly, we have $a \odot v = a, a \odot u < a$ (since $u \notin Y_a$) and $v \odot u = u$ (by Lemma 2.4) while $u \odot v \le u$).

Hence, (X_a, Y_a) is a cut of M. □

We generalize [1, Theorem 3.6] which was proved for linear hoops, and also give a new irreducibility criterion.

Theorem 3.5. *Let M be a linear pseudo hoop. The following statements are equivalent:*

(i) *M is irreducible.*
(ii) *For all $a, b \in M$, $b \to a = a$ implies $b = 1$ or $a = 1$.*
(ii′) *For all $a, b \in M$, $b \rightsquigarrow a = a$ implies $b = 1$ or $a = 1$.*
(iii) *M is a pseudo Wajsberg hoop.*

Proof. (i) ⇒ (ii). Let $b \to a = a$. If $a < 1$ and $b < 1$, by Proposition 3.2, $(X_b^{\to}, Y_b^{\to})$ is a non-trivial cut of M, and by Proposition 3.1 M is not irreducible.

(ii) ⇒ (i). If $M = M_1 \bigoplus M_2$ is a non-trivial decomposition, then for $a \in M_1$ and $b \in M_2$ with $a < b < 1$ we have $b \to a = a > 0$ and $a, b \neq 1$ contradicting (ii). Thus if (ii) holds, there are no non-trivial decompositions.

In the same way we prove (i) ⇔ (ii′).

Now let z be a fixed element of M. For any $x \ge z$, we define $x^{-_z} := x \to z$ and $x^{\sim_z} := x \rightsquigarrow z$. Then (a) $x^{-_z} \odot x = z = x \odot x^{\sim_z}$, (b) $x \le x^{-_z \sim_z}$ and $x \le x^{\sim_z -_z}$, (c) $x^{-_z} = x^{-_z \sim_z -_z}$ and $x^{\sim_z} = x^{\sim_z -_z \sim_z}$, and (d) $(x \odot y)^{-_z} = x \to y^{-_z}$ and $(x \odot y)^{\sim_z} = y \rightsquigarrow x^{\sim_z}$, $x, y \in M$.

To establish (ii) ⇒ (iii), we first prove the following claim.

Claim A. (ii) *implies $x^{-_z \sim_z} = x = x^{\sim_z -_z}$ for any $x \ge z$ and any $z \in M$.*

Step 1. *If, for $z \le y \le x$, $x^{-_z} = y^{-_z}$, then $x = y$.*

Let $x \ge y$. Then $y = y \wedge x = (x \to y) \odot x$ which implies $x^{-_z} = y^{-_z} = (x \to y) \to x^{-_z}$. Hence, $x^{-_z} = 1$ or $x \to y = 1$. Then $x \le z$, i.e., $z = y = x$ or $x \le y$, i.e., $x = y$.

Step 2. *If, for $z \le y \le x$, $x^{\sim_z} = y^{\sim_z}$, then $x = y$.*

It uses the equivalence of (ii) and (ii′), and then it follows the same ideas as the proof of Step 1.

Step 3. *If $x \in M$, then $x^{-_z \sim_z} = x = x^{\sim_z -_z}$.*

It uses properties (ii)–(iii) of ${}^{-_z}$ and ${}^{\sim_z}$ shown above and Step 1 and Step 2 which proves the claim.

(ii) ⇒ (iii).

Now let $x, y \in M$. If $x \le y$, then $(x \to y) \rightsquigarrow y = y$ and by Claim A $(y \to x) \rightsquigarrow x = y^{-_x \sim_x} = y$. If $y \le x$, then $(x \to y) \rightsquigarrow y = x^{-_y \sim_y} = x$ and $(y \to x) \rightsquigarrow x = x$ which proves (W1). In a similar way we prove (W2) which completes that M is a pseudo Wajsberg hoop.

(iii) ⇒ (i). Assume $M = M_1 \bigoplus M_2$, where M_1 and M_2 are linear pseudo hoops which are not trivial. Then there are $y \in M_2 \setminus \{1\}$ and $x \in M_1 \setminus \{1\}$. Check $(x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$ but $(y \to x) \rightsquigarrow x = x \rightsquigarrow x = 1$ which entails M is not Wajsberg. Hence, (iii) implies (i). □

For unbounded linear pseudo hoops we also have the following characterization.

Proposition 3.6. *For an unbounded linear pseudo hoop M, the following statements are equivalent:*

(i) *M is irreducible.*
(ii) *$a = a \odot b$ implies $b = 1$.*
(ii′) *$a = b \odot a$ implies $b = 1$.*

Proof. (i) $\Rightarrow$ (ii). If M is irreducible, M has only trivial idempotents. Assume $a = a \odot b$. By Proposition 3.4, (X_a, Y_a) is a cut which proves that it has to be only trivial, i.e., $b \in Y_a = \{1\}$ or $X_a = \emptyset$.

(i) $\Rightarrow$ (ii). Assume $M = M_1 \bigoplus M_2$. If there exists $a \in M_1 \setminus \{1\}$, for any $b \in M_2$, we have $a = a \odot b$ giving $b = 1$ and $M_2 = \{1\}$. If there exists $b \in M_2 \setminus \{1\}$, then $M_1 \setminus \{1\}$ has to be empty; indeed if $a \in M_1$, $a < 1$, then $a = a \odot b$ which implies $b = 1$, a contradiction. Consequently, M is irreducible.

The equivalence of (i) and (ii′) follows the same ideas as that of (i) and (ii). $\square$

Proposition 3.7. *Let M be a linear Wajsberg pseudo hoop. If M is unbounded, there is a linearly ordered ℓ-group G such that M is isomorphic with the negative cone G^- regarded as a pseudo hoop. If M is bounded, there is a linearly ordered ℓ-group G with strong unit u such that M is isomorphic with the interval* $[0, u]$ *regarded as a pseudo hoop.*

Proof. Suppose that M is unbounded. Then M is cancellative, i.e., if $x \odot z = y \odot z$ and $v \odot a = v \odot b$ then $x = y$ and $a = b$. Indeed, suppose that $x \le y$, then $x = (y \to x) \odot y$ and hence $y \odot z = (y \to x) \odot y \odot z$ which by Proposition 3.6 yields $y \to x = 1$ and $y \le x$. Similarly for the second equality.

Applying [16, Proposition 5.7], there is an ℓ-group G such that M is isomorphic to G^-. It is clear that G is linearly ordered.

If M is bounded, then from the proof of Theorem 3.5 we conclude that M is term-equivalent with a pseudo MV-algebra, and by [7], M is isomorphic to an interval $[0, u]$ of a linear ℓ-group G with strong unit u. $\square$

4. Aglianò–Montagna type decomposition

In this section we show that every linear pseudo hoop can be decomposed in a unique way as an ordinal sum of irreducible linear Wajsberg pseudo hoops. We apply this result to give a new proof that every representable pseudo BL-algebra is good which gives a partial answer to a problem posed in [6, Problem 3.21], and we show that every maximal filter of a linear pseudo hoop is normal. Moreover, we show that every σ-complete linear pseudo hoop is commutative which gives a partial answer to the problem posed in [14] about whether every σ-complete pseudo BL-algebra is commutative.

We follow the ideas of [1]. Let $(I; \le)$ be a linearly ordered set. A subset $J \subseteq I$ is said to be *connected*, if for all $i, j \in J$ and $k \in I$, $i \le k \le j$ implies $k \in J$. A *connected partition* of $(I; \le)$ is a partition of I into connected subsets. A decomposition of a linearly ordered pseudo hoop M is a family $D = \{M_i : i \in I\}$ of linearly ordered pseudo hoops such that $M = \bigoplus_{i \in I} M_i$. Let $\mathcal{D}(M)$ be the set of all decompositions of the linearly ordered pseudo hoop M.

We order $\mathcal{D}(M)$ as follows: if $D = \{M_i : i \in I\}$ and $D' = \{N_j : j \in J\}$, then $D' \preceq D$ if there is a connected partition $\{I_i : j \in J\}$ of I such that

(i) if $j < j'$, then for all $k \in I_j$, $k' \in I_{j'}$ we have $k < k'$;
(ii) $N_j = \bigoplus_{i \in I_j} M_i$.

Then $\preceq$ is a partial ordering on $\mathcal{D}(M)$.

The following result gives a decomposition of any linearly ordered pseudo hoop which was originally proved for linear hoops in [1, Theorem 3.7]. The proof from [1, Theorem 3.7] can be literally used also in our case; to be self-contained, we repeat it here, and, in addition, we add the uniqueness.

Theorem 4.1. *Every linear pseudo hoop (linear pseudo BL-algebra) can be uniquely represented as the ordinal sum of a family of linear pseudo Wajsberg hoops (whose first component is a linear pseudo Wajsberg algebra).*

Proof. Let M be a linearly ordered pseudo hoop and let $(\mathcal{D}(M); \preceq)$ be the poset of its decompositions. Let $\mathcal{C}$ be a chain of decompositions in $\mathcal{D}(M)$. For any $a \in M \setminus \{1\}$ and $D \in \mathcal{C}$, let M^{D_a} be the unique component of D containing a and let $M_a = \bigcap_{D \in \mathcal{D}(M)} M^{D_a}$. Then $M_a \cup \{1\}$ is a pseudo subhoop of M. Now for $a, b \in M$, $M_a \cup \{1\} = M_b \cup \{1\}$ iff a and b lies in the same component of all decompositions in $\mathcal{C}$. Using an axiom of choice we find $I \subseteq M \setminus \{1\}$ such that for every $a \in M \setminus \{1\}$, $I \cap M_a$ contains exactly one element. Then $M = \bigoplus_{a \in I}(M_a \cup \{1\})$, and the decomposition is greater than or equal to any element of the chain $\mathcal{C}$. Applying Zorn's lemma to $(\mathcal{D}(M), \preceq)$, we can find a maximal decomposition of M. Each component of the decomposition must be irreducible and by Theorem 3.5 a linear pseudo Wajsberg hoop (first component is a pseudo Wajsberg algebra if M is).

Uniqueness. Let $\{N_j : j \in J\}$ be a system of linear pseudo Wajsberg hoops such that $M = \bigoplus_{j \in J} N_j$. Given $a \in M \setminus \{1\}$ there are unique M_i and M_j containing a. Let now M_i be an arbitrary component of the decomposition, and assume $a \in M_i \setminus \{1\}$.

Let $b \in M_i \setminus \{1\}$, we assert $b \in N_j$. If not then either $a < b$ or $b < a$. In the first case, there is N_k with $b \in N_k$. We define $X = \bigcup_{j<k}(N_j \setminus \{1\})$ and $Y = \bigcup_{j>k} N_j$. Then (X, Y) is a cut in M and $(X \cap M_i, Y \cap M_i)$ is a cut in M_i such that $a \in X \cap M_i$ and $b \in Y \cap M_i$. By Proposition 3.1, M_i is not irreducible, a contradiction. In a similar way we proceed if $b < a$.

Hence $M_i \subseteq N_j$, and by symmetry, $N_j \subseteq M_i$, i.e., $M_i = N_j$. Therefore, every M_i coincides with some N_j and vice versa. $\square$

We will call the above decomposition of a linear pseudo hoop from Theorem 4.1 an *Aglianò–Montagna type decomposition*.

In view of Proposition 3.7, Theorem 4.1 can be reformulated as follows.

Corollary 4.2. *Every linearly ordered pseudo hoop is the ordinal sum of a system whose each component is either the negative cone of a linear ℓ-group or an interval in a linear unital ℓ-group with strong unit.*

As a corollary, we have that for the variety of pseudo hoops (pseudo BL-algebras) the basic bricks are ℓ-groups with their negative as well as unital ℓ-groups with their intervals, and the basic construction is the ordinal sum.

We say that a pseudo BL-algebra M is *good* if M satisfies the identity

$$x^{-\sim} = x^{\sim -}, \quad x \in M.$$

This notion was introduced [6,14]. We recall that every GMV-algebra is good and also an ordinal sum of GMV-algebras is good.

On the other hand, there is an open problem of whether there exists an example of a pseudo BL-algebra which is not good, [6]. We gave in [9] a partial positive answer to this problem showing that every linearly ordered pseudo BL-algebra (whence every representable pseudo BL-algebra) is good.

We say that a pseudo BL-algebra M is *representable* if it can be represented as a subdirect product of linear pseudo BL-algebras.

Using the Aglianò–Montagna type decomposition, we now give a new proof that every representable pseudo BL-algebra is good, [9], and it gives a partial answer to the problem posed in [6, Problem 3.21] of whether every pseudo BL-algebra is good.

Theorem 4.3. *Every representable pseudo BL-algebra is good.*

Proof. First assume that M is a linearly ordered pseudo BL-algebra. Due to the Aglianò–Montagna type decomposition, Theorem 4.1, $M = \bigoplus_i M_i$, where $\{M_i\}$ is a system of pseudo Wajsberg hoops with the first component, M_0, a pseudo MV-algebra. Since M_0 is good, hence, if $x \in M_0$, then $x^{-\sim} = x = x^{\sim -}$. If $x \in M_i \neq M_0$, then $x^{-\sim} = 1 = x^{\sim -}$, proving M is good.

Let now M be representable via $\{M_t : t \in T\}$, since every M_t is linear, M_t is good, so $\prod_{t \in T} M_t$ is good, and consequently, M is good. $\square$

The last result can be generalized as follows.

Proposition 4.4. *Let M be a linear pseudo hoop and let $z \in M$. For any $x \in M$ with $x \geq z$, we have $x^{-_z \sim_z} = x^{\sim_z -_z}$.*

Proof. We use the Aglianò–Montagna type decomposition of M, i.e., $M = \bigoplus_i M_i$. Since every M_i is a pseudo Wajsberg hoop, due to Claim A in the proof of Theorem 3.5 we have $x^{-_z \sim_z} = x = x^{\sim_z -_z}$ if $x, z \in M_i$. If $z \in M_i$ and $x \in M_j$ for $i < j$, then $x^{-_z \sim_z} = 1 = x^{\sim_z -_z}$. $\square$

In [9, Theorem 4.3], we proved that every maximal filter of a linear pseudo BL-algebra is normal. In what follows, we extend this result for linear pseudo hoops using a different method, Aglianò–Montagna decomposition, and demonstrate the power of this decomposition.

We recall that a filter F of a pseudo hoop is called (i) *maximal* if it is proper and not properly contained in any proper filter of M, and (ii) *normal* if $x \to y \in F \iff x \rightsquigarrow y \in F$ for each $x, y \in M$. It is possible to show that a filter F is normal iff $x \odot F = F \odot x$ (we set $x \odot F = \{x \odot f : f \in F\}$ and $F \odot x = \{f \odot x : f \in F\}$.) We recall that normal filters are in a one-to-one correspondence with congruences.

Theorem 4.5. *Every maximal filter F of a linear pseudo hoop M is normal.*

Proof. If M is linear, then any two filters of M are comparable. If M is bounded, $F = \{x \in M : x^n > 0\}$. If M is unbounded, then F exists iff $F_0 := \bigcup\{F' : F' \text{ is a proper filter of } M\} \neq M$ in such a case $F = F_0$.

To prove that F is normal it is sufficient to verify that if $x > y$ and $x \notin F$, then $x \to y \in F$ iff $x \rightsquigarrow y \in F$.

Express M as an ordinal sum of Aglianò–Montagna type linear Wajsberg pseudo hoops $M = \bigoplus_{i \in I} M_i$. Then there is a linear Wajsberg pseudo hoop M_{i_0} with its maximal filter F_{i_0} such that $F_{i_0} = F \cap M_{i_0}$. In addition, there is no $i \in I$ such that $i < i_0$ and $x, y \in M_i$. Consequently, if M is bounded, M_{i_0} is a linear pseudo MV-algebra, therefore, F_{i_0} is a normal filter by [8, Proposition 5.4].

Assume thus M is unbounded. Now let $z \in M_{i_0} \setminus F_{i_0}$, and define $M^z := \{u \in M_{i_0} : z \le u \le 1\}$. Then M^z is a bounded linear Wajsberg pseudo hoop, i.e., a linear pseudo MV-algebra, with respect to $\odot_z, \to, \rightsquigarrow, z$, and 1, where $u \odot_z v := (u \odot v) \vee z$ ($u, v \in M^z$). Let F^z be its maximal filter. According to [8, Proposition 5.4], F^z is a normal filter of M^z.

Let $u \in F_{i_0}$, then $u^n = u \odot \cdots \odot u > z$ for any $n \ge 1$. Hence $u \odot_z \cdots \odot_z u = u^n \vee z = u^n > z$ which yields $u \in F^z$ for any $z \in M_{i_0} \setminus F_{i_0}$, i.e. $F_{i_0} \subseteq \bigcap\{F^z : z \in M_{i_0} \setminus F_{i_0}\}$. Now choose $w \in \bigcap\{F^z : z \in M_{i_0} \setminus F_{i_0}\}$. Then $w > z$ for any $z \in M_{i_0} \setminus F_{i_0}$, whence $w \in F_{i_0}$ while if not, then $w \in M_{i_0} \setminus F_{i_0}$ and $w > w$, a contradiction. Therefore, $F_{i_0} = \bigcap\{F^z : z \in M_{i_0} \setminus F_{i_0}\}$. Since every F^z is normal, so is F_{i_0} which implies that F is normal. $\square$

The theorem can be extend as follows. Let $g < 1$ be an element of a pseudo hoop M. We say that a *value* of g is a filter V of M such that $g \notin V$, and V is maximal with respect to this property. For any value V of an element g we have that there is a unique least filter V^* properly containing V; it is called a *cover* of V. It is equal to the filter generated by V and the element g.

Corollary 4.6. *Every value of any element $g \in M \setminus \{1\}$ of a linear pseudo hoop M is normal in its cover.*

Proof. Let V be a value of g. Due to linearity of M, V is unique, and $V = \{x \in M : x^n > g \ \forall\, n \ge 1\}$. Set $M_g = \{x \in M : x \ge g^n \text{ for some } n \ge 1\}$. Then $(M_g; \odot, \to, \rightsquigarrow, 1)$ is a linear pseudo hoop, and V is a maximal filter of M_g. By Theorem 4.5, V is normal in M_g. Since M_g is the cover of V in M, V is normal in its cover. $\square$

In [7], we have proved that every σ-complete pseudo MV-algebra is commutative. Georgescu [14, p. 228] asked whether this is true also for pseudo BL-algebra. Using the Aglianò–Montagna type decomposition we give a partial answer to this problem showing that every σ-complete linear pseudo hoop is commutative. We recall that a pseudo hoop M is *σ-complete* if every sequence in M has a supremum in M and every sequence bounded from below in M has an infimum in it. Recently, [20] showed, in particular, that every finite pseudo BL-algebra is commutative.

Theorem 4.7. *Every σ-complete linear pseudo hoop is commutative.*

Proof. Let M be a σ-complete linear pseudo hoop. Express M as an ordinal sum of Aglianò–Montagna type linear Wajsberg pseudo hoops $M = \bigoplus_{i \in I} M_i$. To prove that M is commutative it is sufficient to show that every linear Wajsberg pseudo hoop M_i is commutative.

We claim that every M_i is a σ-complete pseudo hoop. If a sequence from M_i is bounded from below by an element from M_i then it has an infimum in it.

Let $\{a_n\}$ be an arbitrary sequence from M_i, and let $a = \bigvee_n a_n \in M$. If there is $x \in M_i \setminus \{1\}$ such that $x \ge a_n$ for any n, then $a \in M_i$. If 1 is a unique upper bound in M_i of $\{a_n\}$, there are two possibilities. Either 1 is a unique upper bound of $\{a_n\}$ in M, then $1 = a \in M_i$ or there is an element $x \in M \setminus \{1\}$ which is also an upper bound of $\{a_n\}$. Hence, $x \geq a$ and a belongs to a unique M_j with $j > i$, there is no $k \in I$ such that $i < k < j$, and hence, a is the least element of M_j.

In a similar way we can show that if $\{b_n\}$ is another sequence from M_i with $b = \bigvee_n b_n$ and if 1 is a unique upper bound of $\{b_n\}$ in M_i, then $a = b$.

Therefore, 1 can be supposed to be a supremum of $\{a_n\}$ in M_i, that is M_i is σ-complete.

Let z be an arbitrary element in M_i and define $M_i^z := \{x \in M_i : z \leq x \leq 1\}$. As in the proof of Theorem 4.5, M^z is a linear pseudo MV-algebra with respect to $\odot_z, \to, \rightsquigarrow, z$, and 1, where $u \odot_z v := (u \odot v) \vee z$ $(u, v \in M_i^z)$. Since the order in the pseudo MV-algebra M_i^z is the same as that in M_i, we have that M_i^z is a σ-complete pseudo MV-algebra, and by [7, Theorem 4.2], M_i^z is commutative.

Let $x, y \in M_i$ be given. Then either $x \odot y \leq y \odot x$ or $y \odot x \leq y \odot x$. Suppose the first possibility, and set $z := x \odot y$. Then $x \odot_z y = y \odot_z x$, i.e., $x \odot y = (y \odot x) \vee z = (y \odot x) \vee (x \odot y)$ which proves $x \odot y \geq y \odot x$. Hence, every M_i is commutative, and consequently, M is commutative, too. □

As a direct corollary of Theorem 4.7 we have that if a σ-complete pseudo hoop M is a subdirect product of finitely many linear pseudo hoops, $M_1, \ldots, M_k$, then M is commutative. Indeed, every $M_1, \ldots, M_k$ is then σ-complete.

In addition, in view of Corollary 4.2, every σ-complete linear pseudo hoop is an ordinal system whose every component is either the negative cone of a linear Abelian Dedekind σ-complete ℓ-group or an interval of a linear Abelian Dedekind σ-complete ℓ-group with strong unit. As a side-result, we have obtained a known result that there is no non-commutative continuous pseudo t-norm.

It still remains open a general question whether every σ-complete pseudo BL-algebra is commutative.

5. Concluding remarks

(1) Kühr in [21] proved that representable pseudo BL-algebras form a variety, and according to Theorem 4.3, each member of the variety is good. Anyway, there is still an open question about every pseudo BL-algebra is good.

We note that the equational basis for the variety of representable pseudo BL-algebras is by [21] (i) $(y \to x) \vee (z \rightsquigarrow ((x \to y) \odot z)) = 1$, and (ii) $(y \rightsquigarrow x) \vee (z \to (z \odot (x \rightsquigarrow y))) = 1$. It is worth mentioning that the same basis is also that for the variety of representable pseudo hoops. Indeed, if a pseudo hoop M is a subdirect product of linear pseudo hoops, $\{M_t : t \in T\}$, we set $\hat{M}_t = M_t$ if M_t is bounded, otherwise $M_t = \{0, 1\} \bigoplus M_t$. Set $\hat{M} = \prod_{t \in T} \hat{M}_t$. Then $\hat{M}$ is a representable bounded pseudo hoop (= pseudo BL-algebra), with M as its pseudo subhoop, hence (i) and (ii) hold in $\hat{M}$ and as well as in M.

Conversely, let a pseudo hoop M satisfy (i)–(ii). If M is bounded, M is representable by [21]. If M is unbounded, set $\hat{M} = \{0, 1\} \bigoplus M$ obtaining a bounded pseudo hoop for which (i)–(ii) hold. Hence, $\hat{M}$ is representable, say, by $\{M_t : t \in T\}$, and M is a pseudo subhoop of $\hat{M}$. Any $\hat{M}_t := \pi_t(M)$ is a linear subhoop of M_t, and M is representable by $\{\hat{M}_t : t \in T\}$.

(2) We recall that in [9, Theorem 5.1] we obtained a decomposition of a Hájek type of saturated linear pseudo BL-algebra as an ordinal type of irreducible (of the Hájek type) linear pseudo BL-algebras.

In [6, Theorem 3.26], it was proved that any saturated and irreducible (of the Hájek type) linear pseudo BL-algebra M is either a linear pseudo MV-algebra or a linear product pseudo BL-algebra (i.e. a pseudo BL-algebra such that, for all $x, y, z \in M$, (P1) $x \wedge x^- = 0$ or $x \wedge x^\sim = 0$, (P2) $x^= \odot (x \odot z \to y \odot x) \leq x \to y$, and (P3) $x^\approx \odot (z \odot x \rightsquigarrow z \odot y) \leq x \rightsquigarrow y$). Due to Theorem 4.3, the assumption of M to be good is now superfluous, and each component of the Hájek type decomposition in [9, Theorem 5.1] is either a linear pseudo MV-algebra or a linear product pseudo BL-algebra.

(3) As mentioned in [1], these two types of decomposition of a pseudo BL-algebras (even in the commutative case) are different: A linear pseudo BL-algebra M is irreducible in the Hájek sense iff it is either a pseudo BL-algebra or a product pseudo BL-algebra, whereas M is irreducible in the sense of Aglianò–Montagna iff M is a Wajsberg pseudo hoop.

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